Chapter 4

The Finite Volume Method for Diffusion Problems

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Introduction

General transport equation is

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi u) = div(\Gamma\, grad\,\phi) + S_\phi$$

For steady diffusion:

$$div(\Gamma\, grad\,\phi) + S_\phi = 0$$

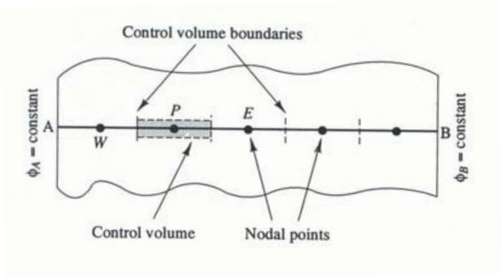
Control volume integration gives

$$\int_{CV} div(\Gamma\, grad\,\phi)dV + \int_{CV} S_\phi dV = \int_A \mathbf{n}\cdot(\Gamma\, grad\,\phi)dA + \int_{CV} S_\phi dV$$

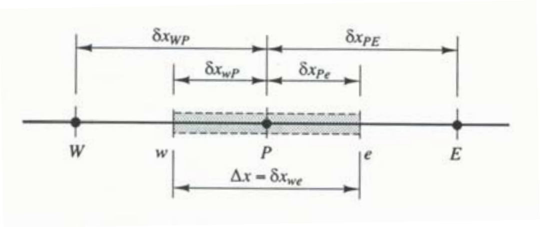
Finite Volume Method for One-dimensional Steady State Diffusion

Steady-state diffusion of a general property ϕ in one-dimensional domain is

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S_\phi=0$$



Step 1: Grid generation:



Step 2: Discretisation

Integration of the diffusion equation over the CV gives

$$\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV+\int_{\Delta V}S_\phi\, dV=\left(\Gamma A\frac{d\phi}{dx}\right)_e-\left(\Gamma A\frac{d\phi}{dx}\right)_w+\bar{S}\Delta V=0$$

To find expressions at the east and west faces, use Taylor series approximations

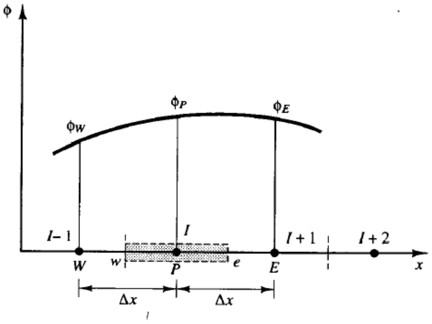
$$\phi(x+\Delta x)=\phi(x)+\left(\frac{\partial\phi}{\partial x}\right)_x\Delta x+\left(\frac{\partial^2\phi}{\partial x^2}\right)_x\frac{\Delta x^2}{2}+\cdots$$

$$\phi(x+\Delta x)=\phi(x)+\left(\frac{\partial\phi}{\partial x}\right)_x\Delta x+\left(\frac{\partial^2\phi}{\partial x^2}\right)_x\frac{\Delta x^2}{2}+\cdots$$

$$\phi_E=\phi_P+\left(\frac{\partial\phi}{\partial x}\right)_P\Delta x+\boxed{\left(\frac{\partial^2\phi}{\partial x^2}\right)_P\frac{\Delta x^2}{2}+\cdots}$$

$$\phi_W=\phi_P-\left(\frac{\partial\phi}{\partial x}\right)_P\Delta x+\boxed{\left(\frac{\partial^2\phi}{\partial x^2}\right)_P\frac{\Delta x^2}{2}+\cdots}$$

Neglect

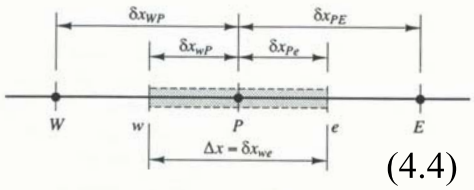


Adding and subtracting →
$$\left(\frac{\partial\phi}{\partial x}\right)_P=\frac{\phi_E-\phi_W}{2\Delta x}$$

At the east face →
$$\left(\frac{\partial\phi}{\partial x}\right)_e=\frac{\phi_E-\phi_P}{\Delta x}$$

Rewriting the diffusion equation for an interior point P:

$$\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV+\int_{\Delta V}S_\phi dV=0$$



$$\left(\Gamma A\frac{d\phi}{dx}\right)_e-\left(\Gamma A\frac{d\phi}{dx}\right)_w+\bar{S}_\phi\Delta V=0 \qquad (4.4)$$

On a uniform grid linear interpolation of Γ is

$$\Gamma_w=\frac{\Gamma_W+\Gamma_E}{2} \qquad \Gamma_e=\frac{\Gamma_P+\Gamma_E}{2} \qquad (4.5)$$

Diffusive flux terms are

$$\left(\Gamma A\frac{d\phi}{dx}\right)_e=\Gamma_e A_e\left(\frac{\phi_E-\phi_P}{\delta x_{PE}}\right) \qquad (4.6)$$

$$\left(\Gamma A\frac{d\phi}{dx}\right)_w=\Gamma_w A_w\left(\frac{\phi_P-\phi_W}{\delta x_{WP}}\right) \qquad (4.7)$$

The source term S may be a function of ϕ → express S in linear form as:

$$S_\phi \Delta V = S_u + S_p \phi_P \qquad (4.8)$$

Substituting (4.6), (4.7) and (4.8) into (4.4)

$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}} \right) + (S_u + S_p \phi_P) = 0 \qquad (4.9)$$

Rearranging,

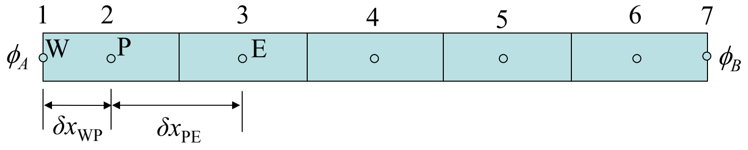
$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w - S_p \right) \phi_P = \left(\frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_E + S_u \qquad (4.10)$$

or,

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u \qquad (4.11)$$

where,

a_W	a_E	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$



Special case: no source terms ($S_\phi = 0$), boundary values ϕ_A, ϕ_B specified

For the point near a west boundary (point 2):

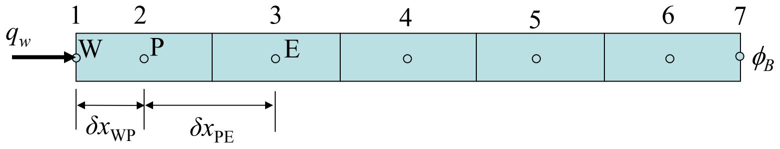
$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}} \right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_A}{\delta x_{WP}} \right) = 0$$

$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_P = 0.\phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e \right) \phi_E + \left(\frac{\Gamma_w}{\delta x_{WP}} A_w \right) \phi_A + S_u$$

or,

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u$$

a_W	a_E	a_P	S_P	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	$-\frac{\Gamma_w A_w}{\delta x_{PW}}$	$\frac{\Gamma_w A_w}{\delta x_{PW}} \phi_A$



No source terms ($S_\phi = 0$), heat flux q_w specified at west boundary.

For the point near a west boundary (point 2):

$$\Gamma_e A_e\left(\frac{\phi_E-\phi_P}{\delta x_{PE}}\right)+q_w A_w=0$$

$$\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)\phi_P = 0.\phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)\phi_E + q_w A_w$$

or,

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$$

a_W	a_E	a_P	S_P	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	0	$q_w A_w$

Summary of Boundary Conditions

For a one-dimensional CV of width Δx with a boundary B:

1) Set coefficient $a_B = 0$

2) Source contributions

(a) Fixed value ϕ_B:

Add: $$S_u = \frac{k_B A_B}{\Delta x/2}\phi_B$$

$$S_p = -\frac{k_B A_B}{\Delta x/2}$$

to the source terms S_u and S_p

(b) Fixed flux q_B:

Add $q_B A_B$ in the form of $S_u + S_p\phi_P$ to the source terms S_u and S_p.

Step 3: Solution of equations

Discretised equations of the form (4.11)

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u \qquad (4\text{-}11)$$

must be set up at each nodal point.

→We obtain a system of linear algebraic equations

Solve the system for ϕ values

→Use any matrix solution method.

e.g. Tri-diagonal matrix algorithm.

Tridiagonal Matrix Algorithm

Consider a system of equations that has a tridiagonal form

$$\begin{aligned}
d_2\phi_2 + c_2\phi_2 &= b_2\\
a_3\phi_2 + d_3\phi_3 + c_3\phi_4 &= b_3\\
a_4\phi_3 + d_4\phi_4 + c_4\phi_5 &= b_4\\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad &.\\
a_i\phi_{i-1} + d_i\phi_i + c_i\phi_{i+1} &= b_i\\
. \quad . \quad . \quad . \quad . \quad &.\\
a_{n-2}\phi_{n-3} + d_{n-2}\phi_{n-2} + c_{n-2}\phi_{n-1} &= b_{n-2}\\
a_{n-1}\phi_{n-2} + d_{n-1}\phi_{n-1} &= b_{n-1}
\end{aligned}$$

In matrix form:

$$\begin{bmatrix}
d_2 & c_2 & & & & & & \\
a_3 & d_3 & c_3 & & & & & \\
 & a_4 & d_4 & c_4 & & & & \\
 & & & & & & & \\
 & & & a_i & d_i & c_i & & \\
 & & & & & & & \\
 & & & & & a_{n-2} & d_{n-2} & c_{n-2} \\
 & & & & & & a_{n-1} & d_{n-1}
\end{bmatrix}
\begin{bmatrix}
\phi_2 \\ \phi_3 \\ \phi_4 \\ \\ \phi_i \\ \\ \phi_{n-2} \\ \phi_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
b_2 \\ b_3 \\ b_4 \\ \\ b_i \\ \\ b_{n-2} \\ b_{n-1}
\end{bmatrix}$$

Subtract (a_3/d_2) times row 1 from row 2 → obtain 0 in a_3 position.

$$\begin{cases} d_3 \leftarrow d_3 - \left(\dfrac{a_3}{d_2}\right)c_2 \\ b_3 \leftarrow b_3 - \left(\dfrac{a_3}{d_2}\right)b_2 \end{cases}$$

Note: c_2 is not altered

In general

$$\begin{cases} d_i \leftarrow d_i - \left(\dfrac{a_i}{d_{i-1}}\right)c_{i-1} \\ b_i \leftarrow b_i - \left(\dfrac{a_i}{d_{i-1}}\right)b_{i-1} \qquad (3 \le i \le n-1) \end{cases}$$

At the end of the forward elimination phase, the form of the system is as follows:

$$\begin{bmatrix} d_2 & c_2 & & & & & & \\ & d_3 & c_3 & & & & & \\ & & d_4 & c_4 & & & & \\ & & & & d_i & c_i & & \\ & & & & & & d_{n-2} & c_{n-2} \\ & & & & & & & d_{n-1} \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_i \\ \phi_{n-2} \\ \phi_{n-1} \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ b_4 \\ b_i \\ b_{n-2} \\ b_{n-1} \end{bmatrix}$$

Of course, the b_i's and d_i's are not as they were at the beginning of this process, but the c_i's are. The back substitution phase solves for ϕ_{n-1}, ϕ_{n-2},…, ϕ_2 as follows:

$$\phi_{n-1} \leftarrow \frac{b_{n-1}}{d_{n-1}}$$

$$\phi_{n-2} \leftarrow \frac{1}{d_{n-2}}(b_{n-2} - c_{n-2}\phi_{n-1})$$

In general

$$\phi_i \leftarrow \frac{1}{d_i}(b_i - c_i\phi_{i+1}) \qquad (i = n-2,\ n-3,\ \ldots\ldots.,\ 2$$

Use single dimensioned arrays → (a_i), (d_i), (c_i), (b_i)

Store the solution in array (ϕ_i).

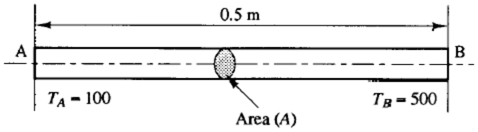
```
subroutine Tri($n$, $a$, $d$, $c$, $b$, $\phi$)
real array $a(n)$, $d(n)$, $c(n)$, $b(n)$, $\phi(n)$
integer $i$, $n$
real $mult$ ! (multiplier)
for $i = 3$ to $n$-1 do
mult $\leftarrow a_i/d_{i-1}$
    $d_i \leftarrow d_i - (mult)c_{i-1}$
    $b_i \leftarrow bi - (mult)b_{i-1}$
end for
$\phi_{n-1} \leftarrow b_{n-1}/d_{n-1}$
for $i = n-2$ to 2 step $-1$ do
    $\phi_i \leftarrow (b_i - c_i\phi_{i+1})/d_i$
end for
end subroutine Tri
```

Worked Examples: 1) One-dimensional steady state diffusion

Governing equation $\rightarrow \dfrac{d}{dx}\left(k\dfrac{dT}{dx}\right) + S = 0$

$k \rightarrow \Gamma$, $\phi \rightarrow T$, S = heat generation per unit volume

Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100 °C and 500 °C respectively. Calculate the steady state temperature in the rod. Take thermal conductivity k = 1000 W/mK, cross-sectional area A = 10×10^{-3}



In this case, S = 0

Solution: Let us divide the rod into 5 equal control volumes (CV's).

Rules for grid generation:

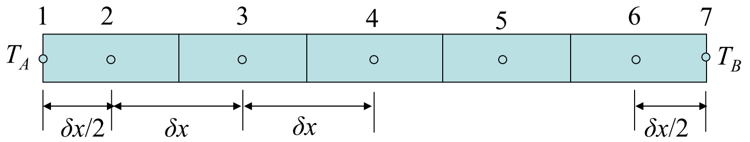
1) Locations of the CV faces are defined first.

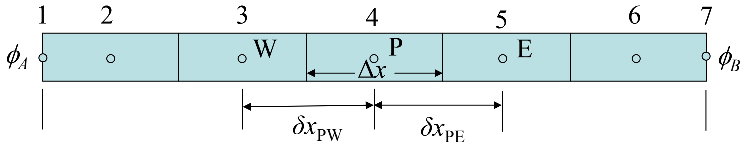
2) Then nodal points are placed at the centers of the CV's.

3) Numbering starts from the boundary node at left.

4) All CV's have a volume of $\delta x.A$

5) Inter-nodal distances are equal to δx, whereas nodal distances for the boundary nodes are equal to $\delta x/2$.





(4.8)

For interior nodes (nodes 3-5):

$$\Gamma_e A_e\left(\frac{\phi_E-\phi_P}{\delta x_{PE}}\right)-\Gamma_w A_w\left(\frac{\phi_P-\phi_W}{\delta x_{WP}}\right)=0 \qquad (4.9)$$

$$\left(\frac{\Gamma_e}{\delta x_{PE}}A_e+\frac{\Gamma_w}{\delta x_{WP}}A_w\right)T_P=\left(\frac{\Gamma_w}{\delta x_{WP}}A_w\right)T_W+\left(\frac{\Gamma_e}{\delta x_{PE}}A_e\right)T_E \qquad (4.10)$$

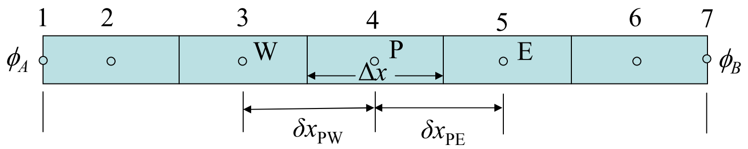
or,

$$a_P T_P = a_W T_W + a_E T_E \qquad (4.11)$$

where,

a_W	a_E	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$

General equation: $a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$ (4-11)



Interior nodes (nodes 3-5):

$S_u = 0,\ S_p = 0,\ \Gamma = k$

a_W	a_E	a_P	S_p	S_u
$\dfrac{\Gamma_w A_w}{\delta x_{WP}}$	$\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	0	0

For boundary node 2:

a_W	a_E	a_P	S_p	S_u
0	$\dfrac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	$-\dfrac{kA}{\Delta x/2}$	$\dfrac{kA}{\Delta x/2}T_A$

For boundary node 6:

a_W	a_E	a_P	S_p	S_u
$\dfrac{\Gamma_w A_w}{\delta x_{WP}}$	0	$a_W + a_E - S_p$	$-\dfrac{kA}{\Delta x/2}$	$\dfrac{kA}{\Delta x/2}T_B$

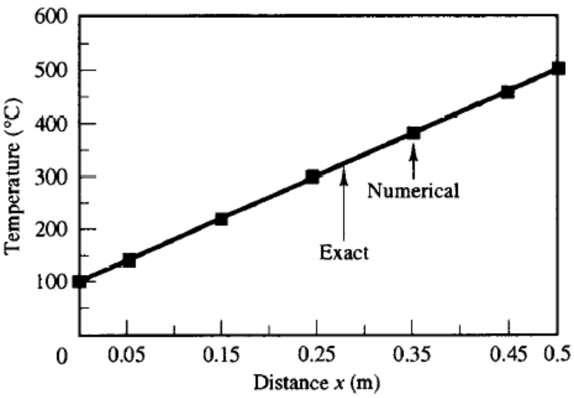
The resulting system of equations are

$$\begin{bmatrix} -a_{P_2} & a_{E_2} & & & & & & \\ a_{W_3} & -a_{p_3} & a_{E_3} & & & & & \\ & a_{W_4} & -a_{p_4} & a_{E_4} & & & & \\ & & & & & & & \\ & & & a_{W_i} & -a_{p_i} & a_{E_i} & & \\ & & & & & & & \\ & & & & & a_{W_{n-2}} & -a_{p_{n-2}} & a_{E_{n-2}} \\ & & & & & & a_{W_{n-1}} & -a_{p_{n-1}} \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ \\ T_i \\ \\ T_{n-2} \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} -Su_2 \\ -Su_3 \\ -Su_4 \\ \\ -Su_i \\ \\ -Su_{n-2} \\ -Su_{n-1} \end{bmatrix}$$

Solve the system of equations using Tri-diagonal matrix algorithm (TDMA) for $T_2, T_3, \ldots T_{n\text{-}1}$, where ($n = 7$)

The solution is:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 100 \\ 140 \\ 220 \\ 300 \\ 380 \\ 460 \\ 500 \end{bmatrix}$$



Comparison of the numerical result with the analytical solution.

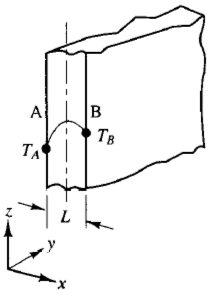
Exact solution is:

$$T = 800x + 100$$

Example 4.2 Now we discuss a problem that includes sources other than those arising from boundary conditions.

Figure 4.6 shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m/K and uniform heat generation $q = 1000$ kW/m^3. The faces A and B are at temperatures of 100 °C and 200 °C respectively. Assuming that the dimensions in the y- and z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + q = 0 \tag{4.25}$$



The governing equation is: $\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{q}=0$

The general equation is: $\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0$

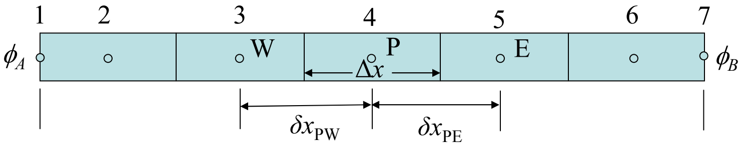
where $S\Delta V = S_u + S_p\phi_P$

Comparing the above equations,

$\phi = T, \Gamma = k,$

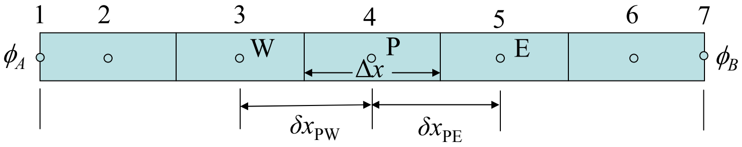
$S_u = q\Delta V \qquad S_p = 0 \qquad$ where, $\Delta V = A\Delta x$

Take area $A = 1$ in the *y-z* plane



Solution is similar to the previous example.

General equation: $a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$ (4-11)



Interior nodes (nodes 3-5):

$S_u = 0, S_p = 0, \Gamma = k$

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	0	$\dot{q}A\Delta x$

For boundary node 2:

a_W	a_E	a_P	S_p	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$	$-\frac{kA}{\Delta x/2}$	$\dot{q}A\Delta x + \frac{kA}{\Delta x/2}T_A$

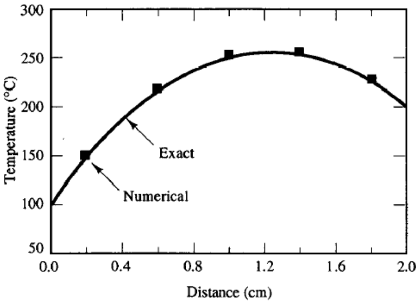
For boundary node 6:

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	0	$a_W + a_E - S_p$	$-\frac{kA}{\Delta x/2}$	$\dot{q}A\Delta x + \frac{kA}{\Delta x/2}T_B$

The solution is:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 100 \\ 150 \\ 218 \\ 254 \\ 258 \\ 230 \\ 200 \end{bmatrix}$$

Node number	2	3	4	5	6
x (m)	0.002	0.006	0.01	0.014	0.018
Finite volume solution	150	218	254	258	230
Exact solution	146	214	250	254	226
Percentage error	2.73	1.86	1.60	1.57	1.76



Comparison of the numerical result with the analytical solution.

Exact solution is: $$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x)\right]x + T_A$$

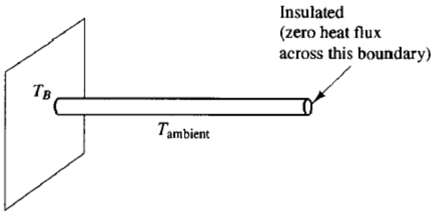
Shown in Figure 4.9 is a cylindrical fin with uniform cross-sectional area A. The base is at a temperature of 100 °C (T_B) and the end is insulated. The fin is exposed to an ambient temperature of 20 °C. One-dimensional heat transfer in this situation is governed by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0 \qquad (4.40)$$

where h is the convective heat transfer coefficient, P the perimeter, k the thermal conductivity of the material and T_∞ the ambient temperature. Calculate the temperature distribution along the fin and compare the results with the analytical solution given by

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(L - x)]}{\cosh(nL)} \qquad (4.41)$$

where $n^2 = hP/(kA)$, L is the length of the fin and x the distance along the fin. Data: $L = 1$ m, $hP/(kA) = 25\ m^{-2}$ (note kA is constant).



The governing equation is: $\frac{d}{dx}\left(kA\frac{dT}{dx}\right)-hP(T-T_\infty)=0$ or $\frac{d}{dx}\left(k\frac{dT}{dx}\right)-n^2(T-T_\infty)=0$

The general equation is: $\frac{d}{dx}(\Gamma\frac{d\phi}{dx})+S_\phi=0$

$$n^2=\frac{hP}{A}$$

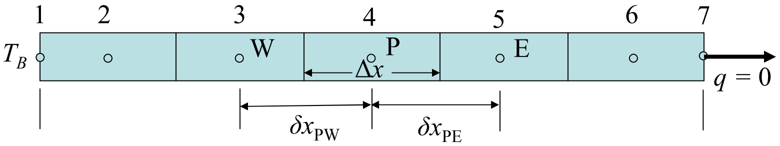
where $S\Delta V=S_u+S_p\phi_P$

$$\Delta V=A\Delta x$$

Comparing the above equations,

$\phi=T, \Gamma=k,$

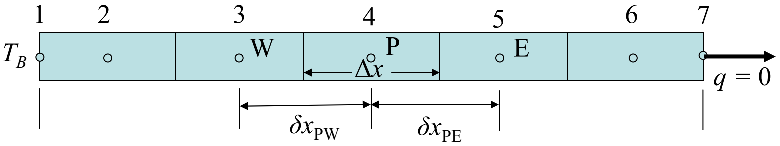
$S_u=n^2T_\infty\Delta x$ $\qquad$ $S_p=-n^2\Delta x$



Solution is similar to the previous example. Find coefficients of

$$a_P\phi_P=a_W\phi_W+a_E\phi_E+S_u$$

General equation: $a_P\phi_P=a_W\phi_W+a_E\phi_E+S_u$ (4-11)



Interior nodes (nodes 3-5):

$S_u=0, S_p=0, \Gamma=k$

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W+a_E-S_p$	$-n^2\Delta x$	$n^2\Delta xT_\infty$

For boundary node 2:

a_W	a_E	a_P	S_p	S_u
0	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W+a_E-S_p$	$-n^2\Delta x-\frac{\Gamma_w A_w}{\Delta x/2}$	$n^2\Delta xT_\infty+\frac{\Gamma_w A_w}{\Delta x/2}T_B$

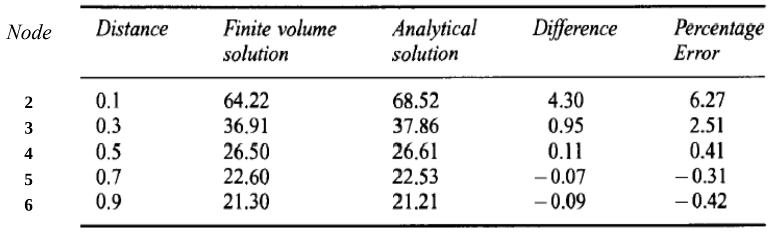
For boundary node 6:

a_W	a_E	a_P	S_p	S_u
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	0	$a_W+a_E-S_p$	$-n^2\Delta x$	$n^2\Delta xT_\infty$

The solution is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 100 \\ 64.22 \\ 36.91 \\ 26.50 \\ 22.60 \\ 21.30 \\ 21.30 \end{bmatrix}$$

Comparison with the analytical solution



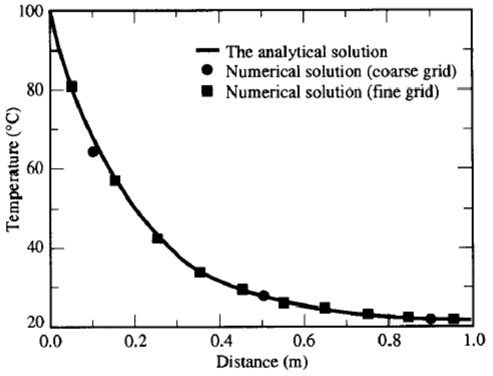
Node	Distance	Finite volume solution	Analytical solution	Difference	Percentage Error
2	0.1	64.22	68.52	4.30	6.27
3	0.3	36.91	37.86	0.95	2.51
4	0.5	26.50	26.61	0.11	0.41
5	0.7	22.60	22.53	−0.07	−0.31
6	0.9	21.30	21.21	−0.09	−0.42

Maximum error: 6.27%

The numerical solution can be improved by employing a finer grid.

Consider the same problem, but use 10 control volumes.

Comparison of the results is given as follows ig. 4.11 Comparison f numerical and alytical results



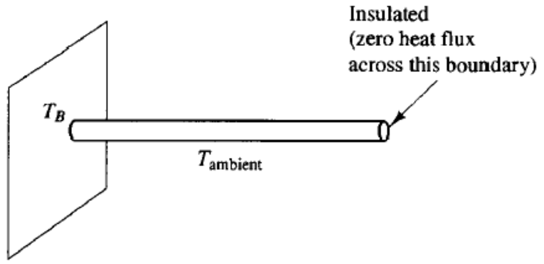
able 4.6

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentag error*
1	0.05	80.59	82.31	1.72	2.08
2	0.15	56.94	57.79	0.85	1.47
3	0.25	42.53	42.93	0.40	0.93
4	0.35	33.74	33.92	0.18	0.53
5	0.45	28.40	28.46	0.06	0.21
6	0.55	25.16	25.17	0.01	0.03
7	0.65	23.21	23.19	−0.02	−0.08
8	0.75	22.06	22.03	−0.03	−0.13
9	0.85	21.47	21.39	−0.08	−0.37
10	0.95	21.13	21.11	−0.02	−0.09

Maximum error: 2.08%

Homework:

Write a fortran program to find the temperature distribution in the rod in example 4.3. Compare the results obtained using 10 and 50 points on a graph.



The geometry of example 4.3.

Pseudo Program to Find a_E Coefficients and Source Terms

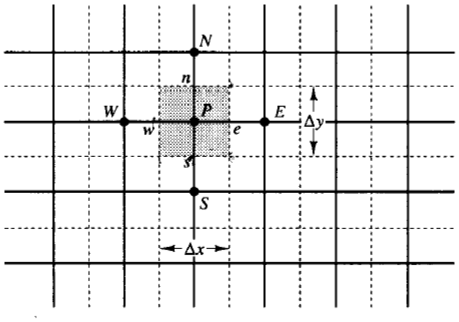
```
Main program
 for $i$ = 2 to $n$-1
Find $\delta x_w(i)$, $\delta x_e(i)$
 end for
call internal_coefficients
call boundary_coefficients
call ap_coefficient
call tdma
end program

Subroutine internal_coefficients
  for $i$ = 2 to n-1
        $a_W(i)=\frac{\Gamma A_w}{\delta x_w(i)}$;   $a_E(i)=\frac{\Gamma A_e}{\delta x_e(i)}$;   $S_p(i)=0,$   $S_u(i)=0$
  end for
end subroutine internal_coefficients)
```

```
Subroutine boundary_coefficients (overwrite on near-boundary coefficients)
  for $i$ = 2 (west boundary)
      $Su(i)=Su(i)+n^2\Delta x T_\infty + a_w(i)T_B$
      $Sp(i)=Sp(i)-n^2\Delta x(i)-a_w(i)$
      $a_w(i)=0$
  end for

   for $i$ = $N$-1 (east boundary)
     no corrections are needed  for $Su$ and $Sp$ since $q_e = 0$ $(Su(i)=Su(i)+q_eA_e)$
     $a_e(i) = 0$
   end for
end subroutine boundary_coefficients

Subroutine a_p coefficient
for $i$ = 2 to $n$-1
$a_p(i) = a_w(i) + a_e(i) - sp(i)$
end for
```

Finite Volume Method for Two-dimensional Diffusion Problems

Consider the two-dimensional steady state diffusion equation

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right)+S=0$$



Integrating the above equation over the CV,

$$\int_{\Delta V}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)dx\cdot dy+\int_{\Delta V}\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}dx\cdot dy\right)+\int_{\Delta V}S_\phi dV=0$$

Noting that $A_e = A_w = \Delta y$ and $A_n = A_s = \Delta x$, we obtain:

$$\left[\Gamma_e A_e\left(\frac{\partial\phi}{\partial x}\right)_e-\Gamma_w A_w\left(\frac{\partial\phi}{\partial x}\right)_w\right]+\left[\Gamma_n A_n\left(\frac{\partial\phi}{\partial y}\right)_n-\Gamma_s A_s\left(\frac{\partial\phi}{\partial x}\right)_s\right]+\bar{S}\Delta V=0 \qquad (4.53)$$

Equation (4.53) represents a balance of the generation of ϕ in a CV and the fluxes through its cell faces

$$\text{Flux across the west face } =\Gamma_w A_w\left.\frac{\partial\phi}{\partial x}\right|_w=\Gamma_w A_w\frac{\phi_P-\phi_W}{\delta x_{WP}}$$

$$\text{Flux across the east face } =\Gamma_e A_e\left.\frac{\partial\phi}{\partial x}\right|_e=\Gamma_e A_e\frac{\phi_E-\phi_P}{\delta x_{PE}}$$

$$\text{Flux across the south face } =\Gamma_s A_s\left.\frac{\partial\phi}{\partial y}\right|_s=\Gamma_s A_s\frac{\phi_P-\phi_S}{\delta y_{SP}}$$

$$\text{Flux across the north face } =\Gamma_n A_n\left.\frac{\partial\phi}{\partial y}\right|_n=\Gamma_n A_n\frac{\phi_N-\phi_P}{\delta y_{PN}}$$

By substitution of the above expressions into eqn. (4.53) we obtain

$$\Gamma_e A_e \frac{\phi_E-\phi_P}{\delta x_{PE}} - \Gamma_w A_w \frac{\phi_P-\phi_W}{\delta x_{WP}} + \Gamma_n A_n \frac{\phi_N-\phi_P}{\delta y_{PN}} - \Gamma_s A_s \frac{\phi_P-\phi_S}{\delta y_{SP}} + \bar{S}\Delta V = 0$$

Substituting the linearised form of the source term $\bar{S}\Delta V = S_u + S_P\phi_P$

$$\left(\frac{\Gamma_w A_w}{\delta x_{WP}} + \frac{\Gamma_e A_e}{\delta x_{PE}} + \frac{\Gamma_s A_s}{\delta y_{SP}} + \frac{\Gamma_n A_n}{\delta y_{PN}} - S_p\right)\phi_P$$

$$= \left(\frac{\Gamma_w A_w}{\delta x_{WP}}\right)\phi_W + \left(\frac{\Gamma_e A_e}{\delta x_{PE}}\right)\phi_E + \left(\frac{\Gamma_s A_s}{\delta y_{SP}}\right)\phi_S + \left(\frac{\Gamma_n A_n}{\delta y_{PN}}\right)\phi_N + S_u$$

This eqn can be written in the form:

$$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + S_u$$

where

$A_w = A_e = \Delta y$
$A_s = A_s = \Delta x$

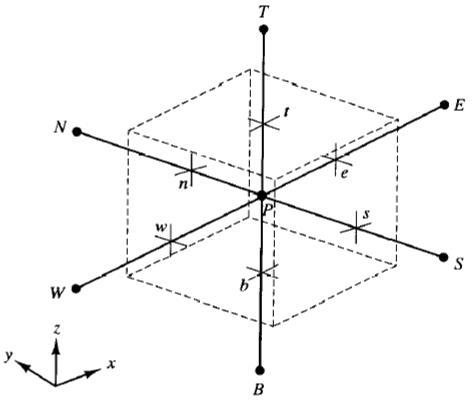
a_W	a_E	a_S	a_N	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$a_W + a_E + a_S + a_N - S_p$

Finite volume method for three-dimensional diffusion problems

Steady state diffusion in a 3D situation is governed by

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial\phi}{\partial z}\right) + S = 0 \qquad (4.58)$$

A typical control volume is shown below.



Integration of eqn (4.58) over the control volume gives

$$\left[\Gamma_e A_e\left(\frac{\partial\phi}{\partial x}\right)_e-\Gamma_w A_w\left(\frac{\partial\phi}{\partial x}\right)_w\right]+\left[\Gamma_n A_n\left(\frac{\partial\phi}{\partial y}\right)_n-\Gamma_s A_s\left(\frac{\partial\phi}{\partial x}\right)_s\right]$$
$$+\left[\Gamma_t A_t\left(\frac{\partial\phi}{\partial z}\right)_t-\Gamma_b A_b\left(\frac{\partial\phi}{\partial z}\right)_b\right]+\bar{S}\Delta V=0$$

which can be discretized as

$$\Gamma_e A_e\frac{\phi_E-\phi_P}{\delta x_{PE}}-\Gamma_w A_w\frac{\phi_P-\phi_W}{\delta x_{WP}}+\Gamma_n A_n\frac{\phi_N-\phi_P}{\delta y_{PN}}-\Gamma_s A_s\frac{\phi_P-\phi_S}{\delta y_{SP}}$$
$$+\Gamma_t A_t\frac{\phi_T-\phi_P}{\delta z_{PT}}-\Gamma_b A_b\frac{\phi_P-\phi_B}{\delta z_{BP}}+(S_u+S_p)\phi_P=0$$

Rearranging

$$a_P\phi_P=a_W\phi_W+a_E\phi_E+a_S\phi_S+a_N\phi_N+S_u$$

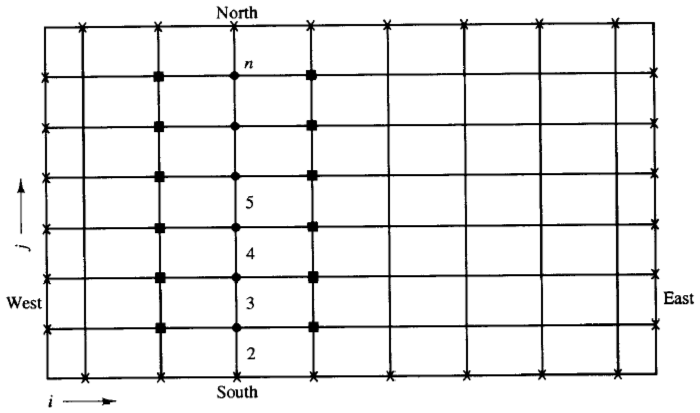
a_W	a_E	a_S	a_N	a_B	a_T	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta z_{BP}}$	$\frac{\Gamma_t A_t}{\delta y_{PT}}$	$a_W+a_E+a_S+a_N$ $+a_B+a_T-S_p$

Summary of discretized equations for diffusion problems

$$a_P\phi_P=\sum a_{nb}\phi_{nb}+S_u$$
$$a_P=\sum a_{nb}-S_p$$
source terms: $\bar{S}\Delta V=S_u+S_p\phi_P$

	a_W	a_E	a_S	a_N	a_B	a_T	a_P
1D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$					$a_W+a_E-S_p$
2D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$			$a_W+a_E+a_S+a_N$ $-S_p$
3D	$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta z_{BP}}$	$\frac{\Gamma_t A_t}{\delta y_{PT}}$	$a_W+a_E+a_S+a_N$ $+a_B+a_T-S_p$

Application of TDMA to two-dimensional problems



● Points at which values are calculated

■ Points at which values are considered to be temporarily known

x Known boundary values

Line by line application of TDMA method

For a point P: $a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + b$

a) Sweep direction: n-s

Rearranging $a_S\phi_S - a_P\phi_P + a_N\phi_N = \boxed{-a_W\phi_W - a_E\phi_E - b}$ → ϕ_W and ϕ_E are known from previous iteration

This is similar to the TDMA equation $a_j\phi_{j-1} + d_j\phi_j + c_j\phi_{j+1} = b_j$

Procedure:

1) Solve along n-s direction along the line for *j* = 2, 3,…, *n*-1 using TDMA
2) Move to the next n-s line
3) Repeat step 1-2.

b) Sweep direction: e-w

For a point P: $a_P\phi_P = a_W\phi_W + a_E\phi_E + a_S\phi_S + a_N\phi_N + b$

Rearranging $a_W\phi_W - a_P\phi_P + a_E\phi_E = \boxed{-a_S\phi_S - a_N\phi_N - b}$ → ϕ_S and ϕ_N are known from previous iteration

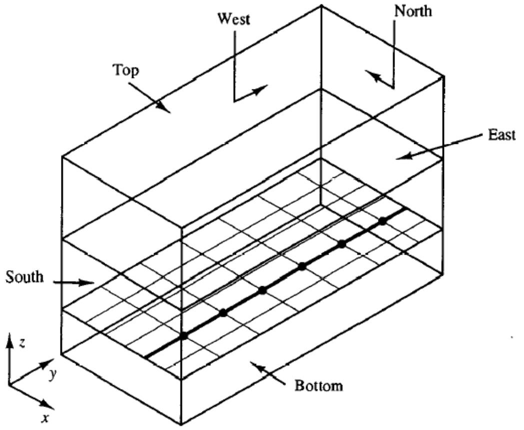
This is similar to the TDMA equation $a_i\phi_{i-1} + d_i\phi_i + c_i\phi_{j+1} = b_i$

Procedure:

1) Solve along e-w direction along the line for $i = 2, 3, \ldots, n-1$ using TDMA
2) Move to the next e-w line
3) Repeat step 1-2.

Application of the TDMA method to three-dimensional problems

For 3-D problems the TDMA method is applied line by line on a selected plane and then the calculation is moved to the next plane.



To solve along n-s direction: $a_S\phi_S - a_P\phi_P + a_N\phi_N = -a_W\phi_W - a_E\phi_E - a_B\phi_B - a_T\phi_T - b$

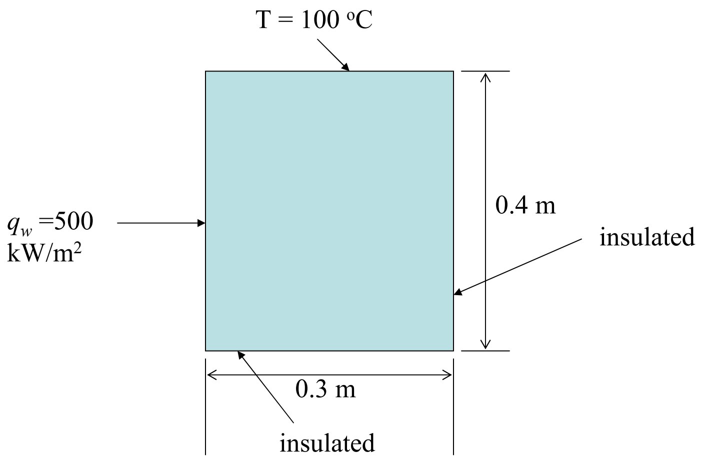
To solve along e-w direction: $a_W\phi_W - a_P\phi_P + a_E\phi_E = -a_S\phi_S - a_N\phi_N - a_B\phi_B - a_T\phi_T - b$

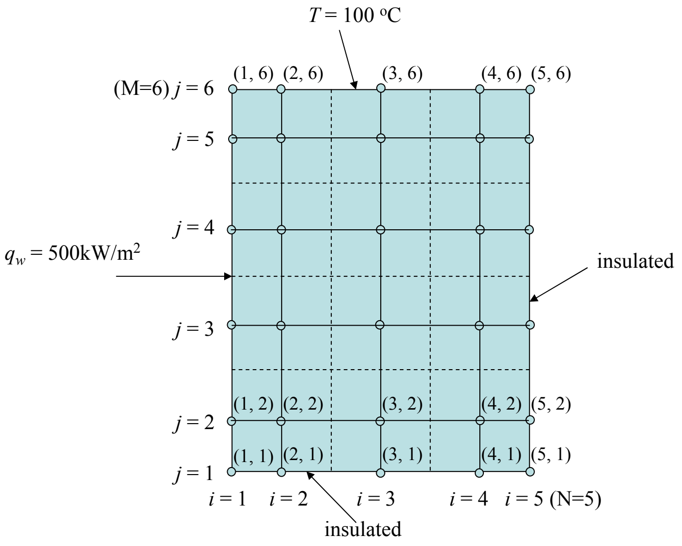
Example:

Consider a 2D plate

Thickness = 1cm, k = 1000W/m/K

Calculate the temperature distribution





First draw control volumes, with equal spacings (dashed lines)

Then, place nodes at the center of the control volumes.

$\Delta x = L_x/N - 2 = 0.3/(5 - 2) = 0.1$, $\Delta y = L_y/M - 2 = 0.4/(6 - 2) = 0.1$

The governing equation is

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial T}{\partial y}\right)=0 \qquad (\Gamma = k)$$

which can be discretised as

$$a_P T_P = a_W T_W + a_E T_E + a_S T_S + a_N T_N + S_u$$

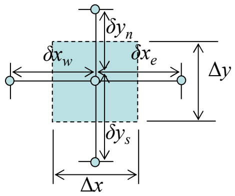
This equation is written for each node (i, j) in the domain

$$\begin{aligned} a_P(i,j)T(i,j) &= a_W(i,j)T(i-1,j) + a_E(i,j)T(i+1,j) \\ &\quad + a_S(i,j)T(i,j-1) + a_N(i,j)T(i,j+1) + S_u(i,j) \end{aligned}$$

For interior points: $(i = 2 – 4, j = 2 – 6)$

$$a_W = \frac{\Gamma A_w}{\delta x_w}; \quad a_E = \frac{\Gamma A_e}{\delta x_e}; \quad a_S = \frac{\Gamma A_s}{\delta x_s}; \quad a_N = \frac{\Gamma A_n}{\delta x_n}$$

$$a_P = a_W + a_E + a_S + a_N - S_p \qquad S_p = 0, \qquad S_u = 0$$



After finding a_P, a_E, a_W, a_N, a_S coefficients follow the following steps

1) Solve the general equation using TDMA along $j = 2$ line (nodes (2, 2, (3, 2), and (4, 2))

$$a_W T_W - a_P T_P + a_E T_E = -a_S T_S - a_N T_N - S_u$$

unknowns

Initially unknown, but set them to zero

→ Temperatures along $j = 2$ are solved (Horizontal sweep)

2) Use TDMA along $j = 3$ line (nodes (2, 3), (3, 3) and (4, 3))

$$a_W T_W - a_P T_P + a_E T_E = -a_S T_S - a_N T_N - S_u$$

Known from previous iteration

Initially was set to zero

3) Repeat until $j = 5$ line

4) Use TDMA along $i = 2$ line

5) Repeat until i =4 line

→ Vertical sweep

6) Go to step 1

7) Repeat steps 1-6 until residual norm becomes $|r| \leq \varepsilon$ where ε = tolerance (use ε = 0.001)

$$|r| = \sum_j \sum_i |r(i,j)|$$

where

$$\begin{aligned} r(i,j) = {} & a_W(i,j)T(i-1,j) + a_E(i,j)T(i+1,j) \\ & + a_S(i,j)T(i,j-1) + a_N(i,j)T(i,j+1) \\ & + S_u(i,j) - a_P(i,j)T(i,j) \end{aligned}$$

Fast Iterative Solvers for Linear Systems of Equations

Apart from TDMA, there are other iterative methods for solving the system of equations. Unlike TDMA, which solves the problem line by line, these iterative methods solves all equations simultaneously. As a result these methods are faster than TDMA. Some of these fast iterative methods are

1) SIP (strongly implicit procedure)

2) MSIP (modified SIP)

3) CG (Conjugate gradient method)

4) BiCGSTAB (bi-conjugate gradient stabilized method)

CG method is used for solving linear systems of equations which have a **symmetric** coefficient matrice. All other methods mentioned above are used for systems of equations involving **non-symmetric** matrices.

Pseudo Program to Solve the 2D Plate Problem

```
Main program
call grids
call internal_coefficients
call boundary_coefficients
call ap_coefficient
for $iter$ = 1 to $iter_{max}$
  call solver
  call residual
  (check if residual is below a desired value)
end for
call print
end program
```

```
subroutine grids
  for $i$ = 2 to $n$-1
    Find $\delta x_w(i)$, $\delta x_e(i)$
  end for
  for $j$ = 2 to $m$-1
    Find $\delta x_s(j)$, $\delta x_n(j)$
  end for
end subroutine grids
```

```
subroutine internal_coefficients
  for $i = 2$ to $n$-1 and $j = 2$ to $m$-1
      $a_W(i,j) = \frac{\Gamma A_w}{\delta x_w(i)}$;   $a_E = \frac{\Gamma A_e}{\delta x_e(i)}$;   $a_S = \frac{\Gamma A_s}{\delta x_s(j)}$;   $a_N = \frac{\Gamma A_n}{\delta x_n(j)}$
      $S_p(i,j) = 0$,   $S_u(i,j) = 0$
  end for
end subroutine internal_coefficients

subroutine boundary_coefficients (overwrite on near-boundary coefficients)
  for $i = 2$ and $j = 2$ to $m$-1 (!west boundary)
    $S_u(i,j) = S_u(i,j) + q_w A_w$
    $a_w(i,j) = 0$
  end for

  for $i = n$-1 and $j = 2$ to $m$-1 (!east boundary)
    $a_e(i,j) = 0$
  end for
```

```
for $i = 2, n$-1 and $j = 2$ (!south boundary)
    $a_s(i,j) = 0$
end for

 for $i = 2, n$-1 and $j = m$-1 (!north boundary)
   Su(i, j) = Su(i, j) +$a_n(i,j)$*$T_{north}$
   Sp(i, j) = Sp(i, j) – $a_n(i,j)$
   $a_n(i,j) = 0$
 end for
end subroutine boundary coefficients

 subroutine ap_coefficient
   for $i = 2$ to $n$-1 and $j = 2$ to $m$-1
      $a_p(i,j) = a_w(i,j) + a_e(i,j) + a_s(i,j) + a_n(i,j) - sp(i,j)$
   end for
 end subroutine ap_coefficient
```

```
Subroutine solver
Call horizontal_sweep
Call vertical_sweep
End subroutine solver

Subroutine horizontal_sweep
Calculate $a(i)$, $d(i)$, $c(i)$, b($i$) for TDMA for solving $x(i)$ values along a horizontal line
call TDMA($a$,$d$,$c$,b, $x$, $n$-2)
end subroutine horizontal_sweep

Subroutine vertical_sweep
Calculate $a(j)$, $d(j)$, $c(j)$, b($j$) for TDMA for solving $x(j)$ values along a vertical line
call TDMA($a$,$d$,$c$,$b$, $x$, $m$-2)
end subroutine vertical_sweep
```